Exercise 1 : Two level system

$$\hat{H} = \frac{\hbar\omega_0}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) - \hat{\vec{d}}\cdot\vec{E}$$

$$= \frac{\hbar\omega_0}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) - \left(d\,\vec{e_z}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\right)\cdot\left(A\,\vec{e_z}\cdot\cos(\omega t - \phi)\right)$$

$$\boxed{\hat{H} = \frac{\hbar\omega_0}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) - dA\cos(\omega t - \phi)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)}$$

1- $\hat{H}_0$ term determines the energy level of the system, ω_0 the resonance frequency determines the energy difference between $|0\rangle$ and $|1\rangle$

$\hat{H}_I$ is the interaction term between drive and system, or how the system transitions from state $|0\rangle$ to $|1\rangle$ and inversly from $|1\rangle$ to $|0\rangle$. It what defines the state manipulation

2-

$$\hat{H}_I = -dA\cos(\omega t - \phi)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$= -\frac{dA}{2}\left(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)}\right)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$\Rightarrow \hat{H} = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\frac{dA}{2}\left(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)}\right) \\ -\frac{dA}{2}\left(e^{i(\omega t - \phi)} + e^{-i(\omega t - \phi)}\right) & \frac{\hbar\omega_0}{2} \end{pmatrix}$$

3 -

Rotating frame with the driving field rotating speed ω:

$$U = e^{i\frac{\omega t}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)}$$

$$\Rightarrow \quad i\hbar \dot{U} U^{+} = i\hbar \cdot i\frac{\omega}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) \cancel{e^{i\frac{\omega t}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)}} \cdot \cancel{e^{-i\frac{\omega t}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)}}$$

$$= -\hbar\frac{\omega}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)$$

$$UHU^{+} = U(H_0 + H_I)U^{+}$$

$$= UH_0U^{+} + UH_IU^{+}$$

$$UH_0U^{+} = \frac{\hbar\omega_0}{2} e^{i\frac{\omega}{2}t(|1\rangle\langle 1| - |0\rangle\langle 0|)}(|1\rangle\langle 1| - |0\rangle\langle 0|) \cdot e^{-i\frac{\omega}{2}t(|1\rangle\langle 1| - |0\rangle\langle 0|)}$$

$$= \frac{\hbar\omega_0}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)$$

$$U H_I U^\dagger = -\frac{dA}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right)\cdot$$

$$\left[e^{i\frac{\omega}{2}t(|1\rangle\langle 1|-|0\rangle\langle 0|)} |0\rangle\langle 1| \, e^{-i\frac{\omega}{2}t(|1\rangle\langle 1|-|0\rangle\langle 0|)} + e^{i\frac{\omega}{2}t(|1\rangle\langle 1|-|0\rangle\langle 0|)} |1\rangle\langle 0| \, e^{-i\frac{\omega}{2}t(|1\rangle\langle 1|-|0\rangle\langle 0|)} \right]$$

$$= -\frac{dA}{2}\left(e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right)\cdot\left[e^{-i\frac{\omega}{2}t} |0\rangle\langle 1| e^{-i\frac{\omega}{2}t} + e^{i\frac{\omega}{2}t} |1\rangle\langle 0| e^{i\frac{\omega}{2}t} \right]$$

$$= -\frac{dA}{2}\cdot\left[e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0| + e^{-2i\omega t+i\phi}|0\rangle\langle 1| + e^{2i\omega t-i\phi}|1\rangle\langle 0| \right]$$

So finally:

$$H_{rot} = i\hbar \dot{U} U^\dagger + U H U^\dagger$$

$$= -\hbar\frac{\omega}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) + \hbar\frac{\omega_0}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) - \frac{dA}{2}\cdot\left[e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0| + e^{-2i\omega t+i\phi}|0\rangle\langle 1| + e^{2i\omega t-i\phi}|1\rangle\langle 0| \right]$$

$$\boxed{H_{rot} = \frac{\hbar\Delta}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) - \frac{\hbar\Omega}{2}\left[e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0| + e^{-2i\omega t+i\phi}|0\rangle\langle 1| + e^{2i\omega t-i\phi}|1\rangle\langle 0| \right]}$$

4 - Applying RWA → neglect fast rotating terms

→ So neglect all terms proportional to exp of $\pm 2\omega$

$\Rightarrow$

$$\hat{H} = \frac{\hbar\Delta}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) - \frac{\hbar\Omega}{2}\left[e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|\right]$$

$$= -\frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & \Delta \end{pmatrix}$$

$$= \frac{\hbar\Delta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

Knowing that $e^{\pm i\phi} = \cos\phi \pm i\sin\phi$

$$\hat{H} = \frac{\hbar\Delta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar\Omega}{2}\left[\begin{pmatrix} 0 & \cos\phi \\ \cos\phi & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -\sin\phi \\ \sin\phi & 0 \end{pmatrix}\right]$$

$$\boxed{\hat{H} = \frac{\hbar\Delta}{2}\sigma_z - \frac{\hbar\Omega}{2}\left(\cos\phi\,\sigma_x + \sin\phi\,\sigma_y\right)}$$

Exercise 1: Time evolution

$$\hat{H} = -\frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & \Delta \end{pmatrix}$$

1 -

Diagonalizing $\hat{H}$:

$$\begin{cases} E_+ + E_- = Tr(\hat{H}) & ① \\ E_+ E_- = det(\hat{H}) & ② \end{cases}$$

① $E_+ + E_- = \frac{\hbar\Delta}{2} - \frac{\hbar\Delta}{2} = 0 \Rightarrow E_+ = -E_-$

② $E_+ E_- = -E_+^2 = -\frac{\hbar^2}{4}(\Delta^2 + \Omega^2)$

$\Rightarrow \quad E_\pm = \pm\frac{\hbar}{2}\sqrt{\Delta^2+\Omega^2}$

$= \pm \hbar/2\ \delta$,

$\rightarrow |+\rangle = \alpha_+ |0\rangle + \beta_+ |1\rangle$

$|-\rangle = \alpha_- |0\rangle + \beta_- |1\rangle$

Finding $|+\rangle$:

$\hat{H}|+\rangle = E_+ |+\rangle$

$$\Leftrightarrow \quad -\frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & -\Delta \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} = \frac{\hbar\delta}{2}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{cases} -\frac{\hbar}{2}\left(\Delta\alpha_+ + \Omega e^{-i\phi}\beta_+\right) = \frac{\hbar\delta}{2}\alpha_+ \\ -\frac{\hbar}{2}\left(\Omega e^{i\phi}\alpha_+ - \Delta\beta_+\right) = \frac{\hbar\delta}{2}\beta_+ \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \Delta\alpha_+ + \Omega e^{-i\phi}\beta_+ = -\delta\alpha_+ \\ \Omega e^{i\phi}\alpha_+ - \Delta\beta_+ = -\delta\beta_+ \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \alpha_+(\Delta + \delta) = -\Omega e^{-i\phi}\beta_+ \\ -\Omega e^{i\phi}\alpha_+ = \beta_+(\delta - \Delta) \end{cases}$$

Fix $\alpha_+ = 1$ so that $\beta_+ = \dfrac{-\Omega e^{i\phi}}{\delta - \Delta}$

$$\Rightarrow |\tilde{+}\rangle = \begin{pmatrix} 1 \\ \dfrac{-\Omega e^{i\phi}}{\delta - \Delta} \end{pmatrix} = \begin{pmatrix} \delta - \Delta \\ -\Omega e^{i\phi} \end{pmatrix}$$

Now normalize the vector

$$\langle\tilde{+}|\tilde{+}\rangle = (\delta - \Delta)^2 + \Omega^2$$

$$= \delta^2 - 2\delta\Delta + \Delta^2 + \Omega^2$$

$$= 2\delta(\delta - \Delta)$$

$$\Rightarrow \quad |+\rangle = \frac{|\tilde{f}\rangle}{\sqrt{\langle \tilde{f}|\tilde{f}\rangle}} = \frac{1}{\sqrt{2\delta(\delta-\Delta)}} \begin{pmatrix} \delta-\Delta \\ -\Omega e^{-i\phi} \end{pmatrix}$$

Finding $|-\rangle$:

$$H|-\rangle = E_-|-\rangle$$

$$-\frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & -\Delta \end{pmatrix}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} = -\frac{\hbar\delta}{2}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix}$$

... Same reasoning as with $|+\rangle$

$$|-\rangle = \frac{1}{\sqrt{2\delta(\delta-\Delta)}}\begin{pmatrix} \Omega \\ e^{i\phi}(\delta-\Delta) \end{pmatrix}$$

2-

$$|\psi(t)\rangle = U(t)\,|\psi_0\rangle$$

$$= e^{-\frac{i\hat{H}t}{\hbar}}\,|0\rangle$$

$$= e^{-\frac{iE_+t}{\hbar}}\,|+\rangle\langle+|0\rangle + e^{-i\frac{E_-t}{\hbar}}\,|-\rangle\langle-|0\rangle$$

$$= e^{-i\frac{\delta t}{2}}\cdot\frac{\delta-\Delta}{\sqrt{2\delta(\delta-\Delta)}}\,|+\rangle + e^{i\frac{\delta t}{2}}\,\frac{\Omega}{\sqrt{2\delta(\delta-\Delta)}}\,|-\rangle$$

$$\langle 1|\psi(t)\rangle = e^{-i\frac{\delta t}{2}}\,\frac{\delta-\Delta}{\sqrt{2\delta(\delta-\Delta)}}\,\langle 1|+\rangle + e^{i\frac{\delta t}{2}}\,\frac{\Omega}{\sqrt{2\delta(\delta-\Delta)}}\,\langle 1|-\rangle$$

$$= e^{-i\frac{\delta t}{2}}\,\frac{(\cancel{\delta-\Delta})\,(-\Omega e^{-i\phi})}{2\delta\,(\cancel{\delta-\Delta})} + e^{i\frac{\delta t}{2}}\,\frac{\Omega\cdot e^{i\phi}\,(\cancel{\delta-\Delta})}{2\delta\,(\cancel{\delta-\Delta})}$$

$$= -e^{-i\frac{\delta t}{2}}\,\frac{\Omega e^{-i\phi}}{2\delta} + e^{i\frac{\delta t}{2}}\,\frac{\Omega e^{i\phi}}{2\delta}$$

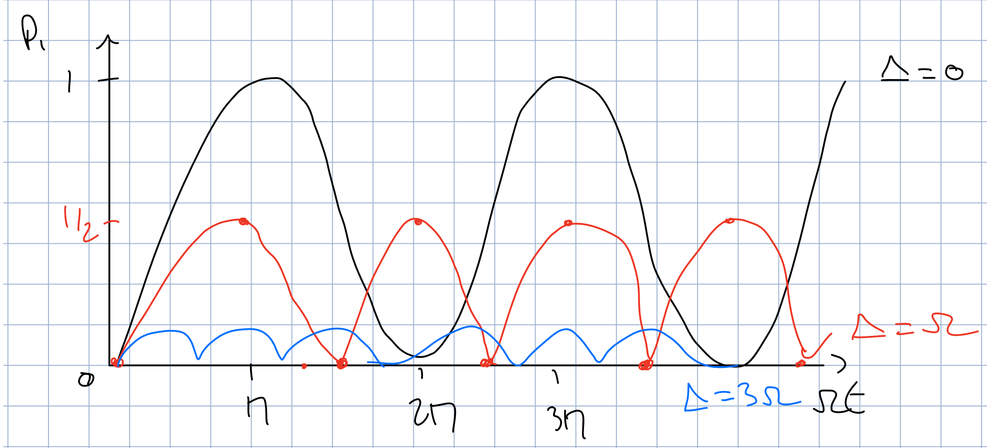
$$= \frac{\Omega}{2\delta}\left(e^{i\frac{\delta t}{2}+i\phi} - e^{-i\frac{\delta t}{2}-i\phi}\right)$$

$$= \frac{\Omega}{2\delta}\cdot 2i\sin\left(\frac{\delta t}{2}+\phi\right)$$

$$\Rightarrow \quad \boxed{\langle 1|\psi(t)\rangle = i\,\frac{\Omega}{\delta}\,\sin\left(\frac{\delta t}{2}+\phi\right)}$$

$$\boxed{P_1(t) = |\langle 1|\psi(t)\rangle|^2 = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\sigma t}{2}+\Phi\right)}$$

Plots



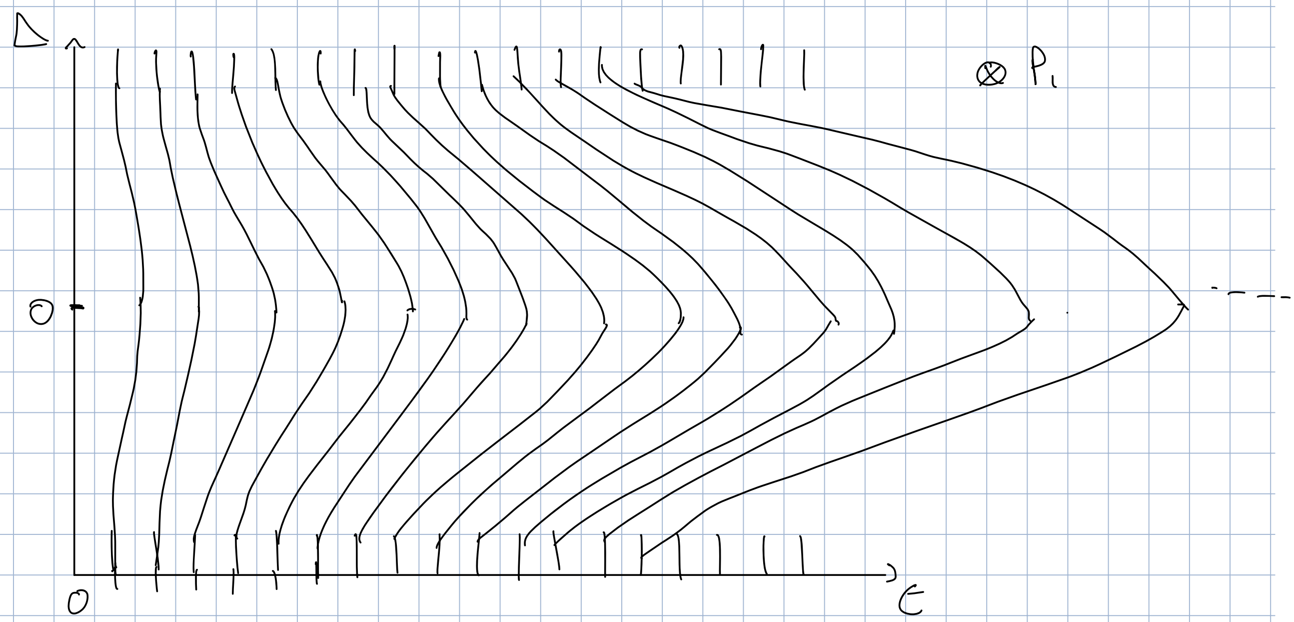
3 -

By fixing the detuning Δ, one can control how much time the driving field is on (this is named a pulse).

So if $\Delta = 0$, a π pulse brings the initial state $|0\rangle$ to the state $|1\rangle$. Looking at $P_1(t)$, we need to send a driving pulse such that $\Omega t = \pi \Rightarrow t = \frac{\pi}{\Omega}$

Same thing for a $\pi/2$ pulse bringing $|0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ or a probability of being in $|1\rangle$: $P_1(t) = 1/2$. We need to send a pulse of length $\Omega t = \pi/2 \Rightarrow t = \frac{\pi}{2\Omega}$

4.



1 – σ_z : rotation around the z axis

σ_x : " " " x axis

σ_y : " " " y axis

You can check by taking a simple Hamiltonian (e.g. $\hat{H} = \hbar\frac{\omega}{2}\sigma_z$) and seeing the time evolution on the arbitrary state

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$$

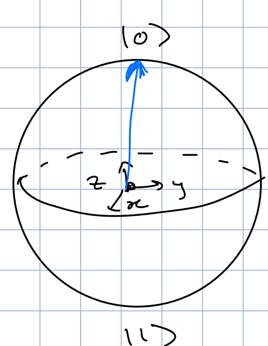
$$e^{-i\frac{\omega t}{2}\sigma_z}|\psi\rangle = \cos\theta/2\,|0\rangle + e^{-i(\omega t-\phi)}\sin\theta/2\,|1\rangle$$

So change in angle $\phi \Rightarrow$ rotation around z.

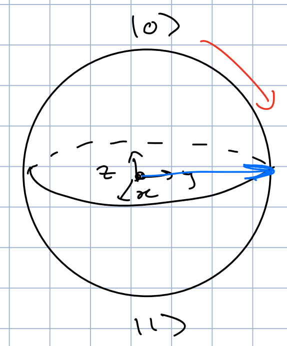
(Use identity

$$e^{i\frac{\theta}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)} = \cos\theta/2\,\mathbb{1} + i\sin\theta/2\,(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$$
)

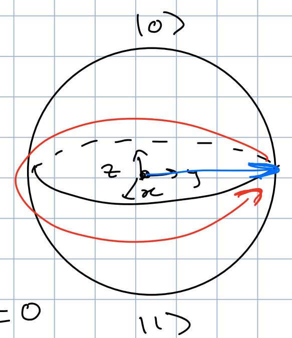
2 –



a)

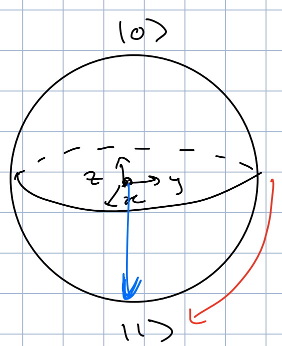


b)

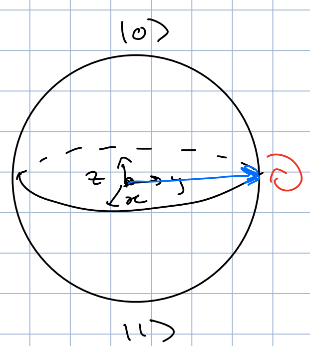


c)

$\phi = 0$



$\phi = \pi/2$



$\phi = \pi$